Mathematics

Start each problem at the top of a new page, please.

Resources: "Fundamentum"
Calculator TI-83 or TI-84

Grading: For each problem the maximally attainable number of points is stated. You do not have to reach the maximum number of points possible in order to get the grade 6.

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1. Given is the function f: 12 points

$$y = f(x) = x^3 - 6x^2 + \frac{45}{4}x \ .$$

a) Discuss the graph of f with regard to zeros, extremal points and points of inflexion. Draw the graph.

b) Find the point P(u, f(u)) on the graph of f under the following conditions: P is different from the origin and the tangent to the graph of f in P passes through the origin.

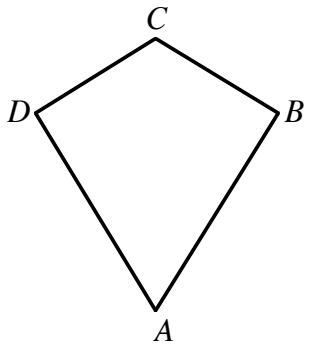
c) Calculate the area bounded by the graph of f and the tangent found in b) lying in the first quadrant.

Alternative for part c):

In case you were unable to solve part b), calculate the area bounded by the graph of f and the line g passing through the origin O and the maximum point of the graph of f.

2. A kite is a quadrilateral which has two pairs of adjacent sides of the same length.
All four vertices lie in the same plane. **12 points**

Consider a kite ABCD with $\overline{AB} = \overline{AD}$ and $\overline{BC} = \overline{CD}$ and given vertices $B(-2, 1, 9)$ and $D(6, 5, 1)$ (cf. diagram).



a) The vertex A of this quadrilateral lies on the line

$$g: \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ 8 \end{pmatrix} + t \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$ Find A.

In case you were unable to solve a), use $A(-6, 11, 1)$.

b) The vertex C lies in the plane E: $3x - 4y + 2z = 20$. Find C.

c) Calculate the area of the kite.

d) Calculate the Cartesian equation of the kite's plane F.

e) The kite ABCD is rotated about the axis AC through 90°.
The rotated figure is the kite AB*CD*. Find B* and D*.

3. The graph of the function with the equation **8 points**

$$y = f(x) = a \cdot e^{2x} + b$$

passes through the points O(0, 0) and $P(\frac{1}{2}, e)$.

a) Find a and b (exact values).

b) Find the angle of intersection between the graph of f and the x-axis.

c) O and P are opposite vertices of a rectangle with sides parallel to the axes.
At which ratio is the rectangle's area partitioned by the graph of f?
(Exact value)

4. A deck of playing cards contains red, blue, green and yellow cards. 12 points

There are cards with the digits 0 to 9 for each colour.
That means the whole deck contains 40 cards.

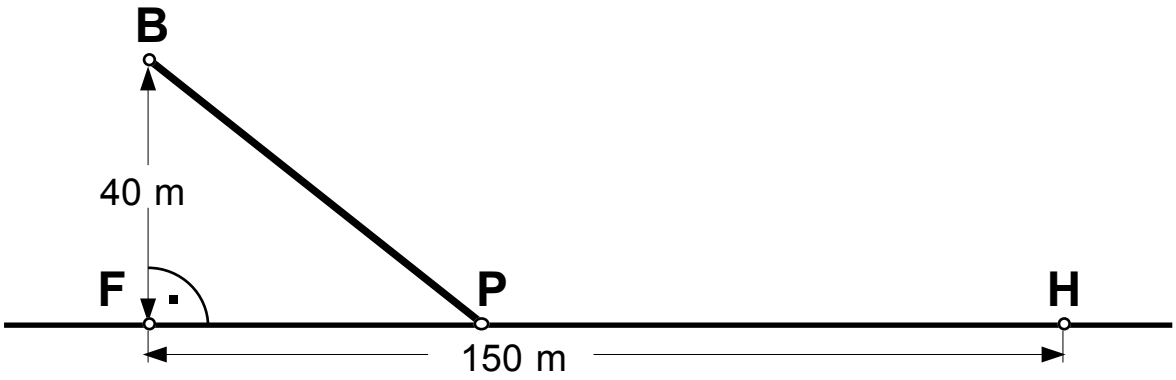
4.1. You draw two cards at random. It is a red and a blue one.
What is the probability that the two cards show the same digit?

4.2. You choose four cards out of the complete deck and put them next to each other on the table. Calculate the probabilities of the following events:
a) The chosen cards show the same colour.
b) The chosen cards show four different colours.
c) Exactly two of the chosen cards are blue.
d) The chosen cards show different colours, but the same digit.

4.3. Somebody proposes the following game: You are allowed to draw cards without returning them to the deck. As soon as one colour comes up for the second time, the game is over. In case you were able to predict the correct number of draws, you win 20 SFr. The stake is 5 SFr. and has to be paid in advance.
What number of draws should you predict? Under these circumstances, what is the average winning amount in this game?

5. A water main (Wasserleitung) has to be constructed that starts at a well B and ends at the nearby farmhouse H. 8 points

This farmhouse is located at a road. The distance of the well to the road equals $\overline{BF} = 40\,\text{m}$. Furthermore the distance $\overline{FH} = 150\,\text{m}$ is known (cf. diagram).



The farmer decides to construct a first straight part of the water main that leads diagonally across the field and ends at the point P and a second part alongside the road starting at P and ending at the farmhouse H.
To lay a water main across the field costs 60 SFr. per meter, to lay it alongside the road costs 50 SFr. per meter.

The total construction should be as cheap as possible. Calculate the distance between the point P and the farmhouse H.