

Mathematics

Please **start solving each problem at the top of a new page!**

Duration: 4 hours

Resources: „Fundamentum“, English dictionary,
Calculator TI-83, TI83+, TI84+, TI-nSpire (**without** CAS)

Grading: The maximum number of points attainable in each problem is stated.
In order to reach grade 6, you do **not** have to get the maximum possible number of points.

1. Consider the function f

12 points

$$f(x) = \frac{1}{4}(x-2)^3 = \frac{1}{4}(x^3 - 6x^2 + 12x - 8)$$

a) Discuss the graph of f with regard to zeros, extremal points and points of inflexion.

b) P is the point of intersection of the graph of f with the y-axis. The tangent to the graph of f in P, the x-axis and the y-axis enclose a triangle.

Calculate its area.

c) The graph of f , the x-axis and the y-axis enclose an area A. It is partitioned in two parts by the tangent from problem b). Calculate the ratio of these parts.

d) (Can be solved without a), b) or c).)

P is the point of intersection of the graph of f with the y-axis. The line g passes through P and touches the graph of f in a point B (different from P). Find the coordinates of B.

2. Given are the four points $A(3 / -1 / 5)$, $B(4 / y_B / 1)$, $C(2 / 1 / z_C)$ and $D(7 / -2 / 16)$ as well as the plane $\mathbb{E}: 4x + 5y + 6z = k$.

10 points

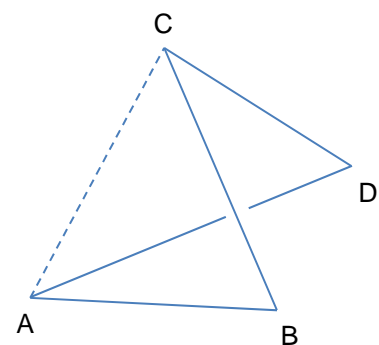
a) The points A, B and C lie in the plane \mathbb{E} . Find k , y_B and z_C .

Now the quadrilateral ABCD is considered.

b) Show that the quadrilateral is not plane.

c) Calculate the folding angle of the quadrilateral ABCD (angle between the triangles ABC and ACD).

d) The point D is reflected in the plane \mathbb{E} . Calculate the coordinates of the mirror image D^* .



3. “Have you ever taken something from a shop without paying?” 12 points
 – When the question above is posed to a group of people, the answers will seldom be truthful. The participants of such a survey will not want to admit to their experience with shoplifting.

Such inhibitions can be overcome by using the following method:

Each participant spins a wheel of fortune in private, without anybody else being able to see the outcome. He or she **must** then follow the instructions given by the wheel. The wheel of fortune will show one of the following two instructions:

<i>Instruction</i>	<i>Tell the truth</i>	<i>Say the opposite</i>
Probability of occurrence	2/3	1/3

- a) Calculate the probability that all of five participants are telling the truth.
 b) What is the probability that, within a group of five participants, of which two have stolen before, nobody admits to stealing?

For the questions c) and d), the percentage of participants who have stolen is assumed to be 13%.

- c) A participant is selected at random. What is the probability of this participant responding, “Yes, I have stolen.“?
 d) What is the probability that a person who spins the wheel and claims to have stolen really has stolen?

For the last question, the fraction p of participants who have stolen is not known.

- e) After questioning a very large number of people, the probability of receiving the answer “yes” is estimated to be 0.39. Calculate the theft rate p .

4. Given is the function $f(x) = 3 \cdot \cos(x)$.

8 points

The letter **T** is inscribed in an arc of the graph of f lying above the x -axis. It consists of a horizontal line segment with length l , whose end points lie on the graph of f , and a vertical line segment with height h .

What value of l produces the **T**'s smallest total length $g = l + h$?

What value produces the largest total length? Calculate the total length g (calculator results) for both cases.

5. Given is the function $f: y = \begin{cases} -x^2 + c & , x < 1 \\ a \cdot e^{-x} & , x \geq 1 \end{cases}$.

12 points

- a) f can be differentiated at the point where the two pieces of the function meet. (In this point the curve is continuous i. e. without jump and smooth i. e. without kink.) Find the parameters a and c .

(In case you were not able to solve a), continue your calculations in the following problems with the **alternative** function $f_1 \dots$)

$$f_1: y = \begin{cases} -1.5 \cdot x^2 + 4.5 & , x < 1 \\ 3e \cdot e^{-x} & , x \geq 1 \end{cases}$$

- b) Calculate the equation of the tangent at the point where the two pieces of the function meet.
- c) Find the content of the area which extends indefinitely to the right and is bounded by the graph of f and the x -axis.

6. The four points $T_1(0 / 0 / 10)$, $T_2(40 / 0 / 10)$, $T_3(40 / 20 / 10)$ and $T_4(0 / 20 / 10)$ are the vertices of a rectangular table.

13 points

- a) Determine the Cartesian equation of the table's plane \mathbb{E} .

A tilted rectangular board is placed on the table. Two adjoining corners P and Q of the board lie on different edges of the table. The board lies in the plane

$$\mathbb{G}: 5x + y + 13z - 190 = 0.$$

- b) Find the coordinates of the points P and Q .

A spherical ball is placed at the point $A(5 / 9 / 12)$ lying on the board and rolls down the board. (The radius of the ball is so small that it can be neglected.)

- c) At which point R does the ball meet the table?

(In case you were not able to solve c), assume that the ball started at $A_1(-4 / 2 / 16)$ and met the table at $R_1(11 / 5 / 10)$.)

The ball continues rolling on the table.

- d) At which point does it fall off the table?

