Mathematics

Start each problem at the top of a new page, please.
Round your results to 4 significant digits.

Time: 4 hours
Resources: „Fundamentum“
Calculator TI-83, TI83+ or TI84+
Grading: For each problem, the maximally attainable number of points is stated. You do not have to reach the maximum number of points possible in order to get the grade 6.

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1. Given are the functions **15 points**

$$y = f(x) = x^2 \text{ and } y = g(x) = \sqrt{x}$$

a) Find the points of intersection of the graphs of the two functions.

b) Determine the angle of intersection of the two graphs in each of the points of intersection.

c) Calculate the area of the finite region between the graphs of f and g.

d) Together with the x-axis and the vertical line with the equation $x = b$, each of these graphs bounds a region. For which value of b are the areas of the two regions equal?

e) Reflect the graph of g at the mirror line with the equation $x = 2$. Find the equation of the function h of the resulting curve. Describe the steps of your solution in words.

f) (Can be solved without using the result of e)!)

The region bounded by the graphs of g, h and the x-axis is rotated about the x-axis to give a solid of revolution. Calculate its volume.

2. Given are the two lines **12 points**

g: $A(1/5/-3)$, $B(2/3/0)$ and h: $C(2/3/3)$, $D(3/1/5)$.

a) Show that the two lines intersect. Calculate the point of intersection.

The points G and H move with constant velocity on the lines g and h.

At the time $t = 0$ the point G is at A, at the time $t = 1$ it is at point B.

At the time $t = 0$ the point H is at C, at the time $t = 1$ it is at point D.

b) Determine the coordinates of the points G and H at the time $t = 2$.
Calculate the distance between G and H at the time $t = 10$.

c) At what time t does the distance between G and H equal 3?

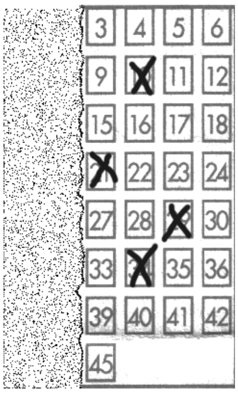
d) At what time t is the distance between G and H the shortest possible?
Calculate this distance.

3. (For this problem you may use the calculator without any restrictions.) **10 points**

When Quentin wanted to pass his e-mail address to a new party acquaintance, there was no time to waste: he tore off a scrap of the first piece of paper at hand.

Later on, that piece of paper turned out to be his lottery coupon, which he had already filled in with six crosses.

The lottery coupon, which at first showed all the numbers from 1 to 45, now looked like this:



a) Quentin has forgotten where the missing two crosses were placed. How many possibilities are imaginable?

b) What is the probability that the lotto draw yields precisely the four numbers crossed in the picture, along with two numbers from the missing part of the coupon?

c) Quentin hears about the winning numbers and compares them to the remains of his ticket as shown above: Everything is correct in the part that is still visible!

   Calculate the average gain he may expect if he gets CHF 50 for 4, CHF 2500 for 5 and CHF 900000 for 6 matches (and if he finds the receipt he got at the kiosk...)

d) The statistics of Swisslos mentions the number that has appeared in the fewest lotto draws so far. At the moment, that number is „14“. It has only been drawn 192 times out of 1635.

   Calculate how many times any number can be expected on average to appear in 1635 lotto draws.

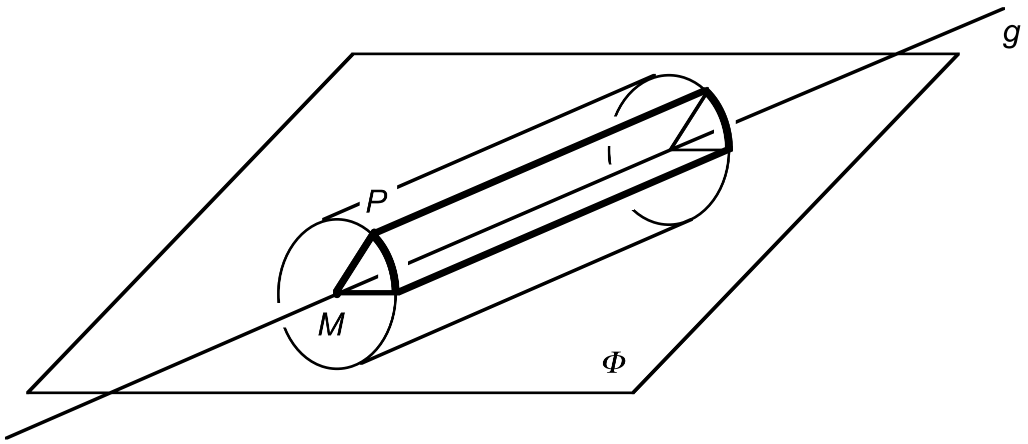
   Furthermore, calculate the probability that the number „14“ appears as few times or even less.

4. Given are the plane Φ: $2x - 2y + z + 10 = 0$, 12 points

the point $P(3/1/4)$ and the line g: $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -10 \end{pmatrix} + t \cdot \begin{pmatrix} 7 \\ 2 \\ -10 \end{pmatrix}$.

a) Show that the line g lies in the plane Φ.

b) The line g is the axis of a right circular cylinder. The point P lies on the cylinder's base circle:



Calculate the coordinates of the circle's midpoint M.

c) The height of the cylinder is $h = 18$. Consider the cylinder sector marked in the sketch. One of the straight line edges of the cylinder sector's curved surface lies in Φ, the other is parallel to the aforementioned line and passes through P. Calculate the sector's volume.

5. The functions $y = f_k(x) = e^{kx} + e^{-x}$ $(k > 0)$ are considered. 15 points

a) In the case of $k = 2$, discuss the graph of f_2 with regard to zeros, maxima and minima. Draw the graph of f_2.

b) The point $P(0/?)$ lies on the graph of f_k. For which k does the tangent to the graph of f_k in P pass through the point $Q(-1/0)$?

c) Calculate the area A_k of the region bounded by the graph of f_k, the line g_k: $y = -\frac{2}{k} \cdot x$, the y-axis and the vertical line with the equation $x = 1$
(as a function of k).
For which k is the area as small as possible?
(You do not have to show that A_k is a minimum.)